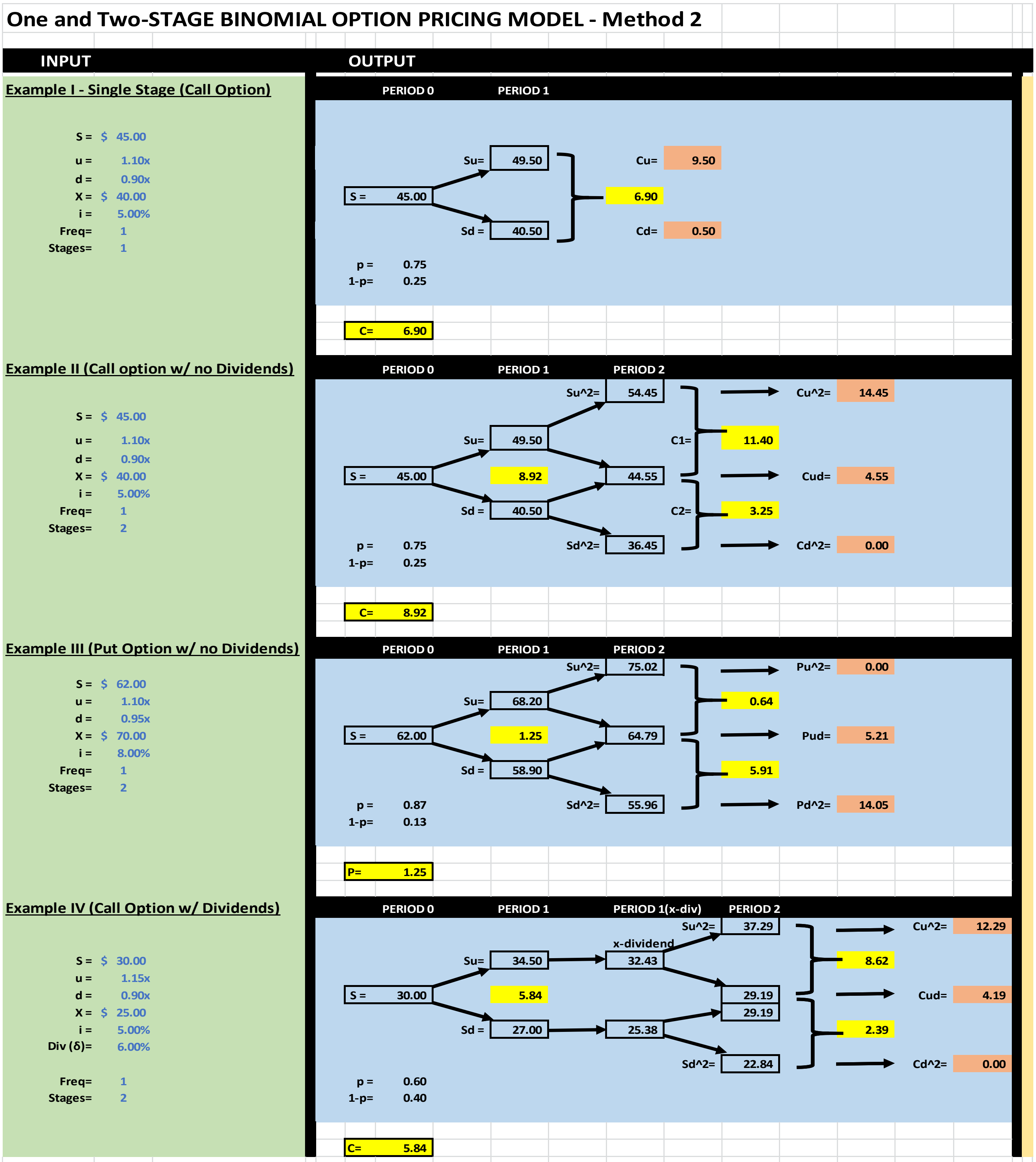
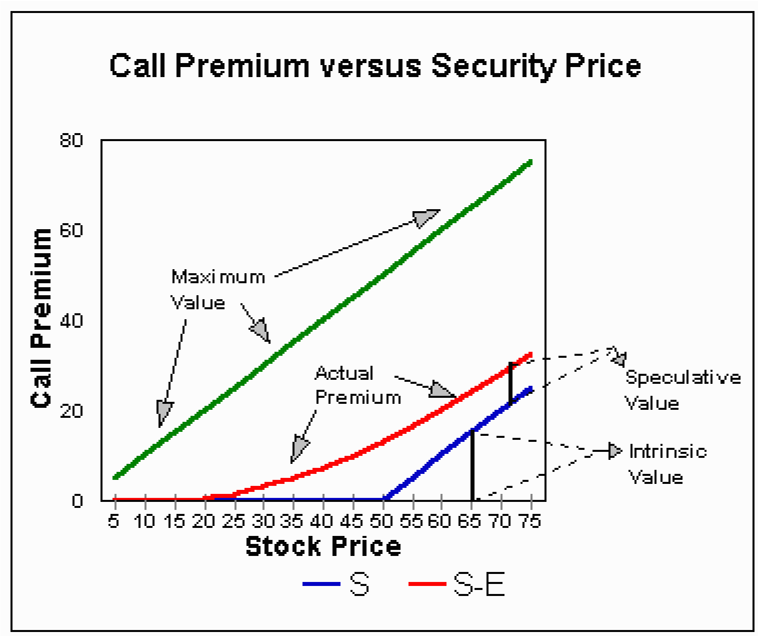
LECTURE 9


One and Two-STAGE BINOMIAL OPTION PRICING MODEL - Method 2
INPUT
OUTPUT
Example I - Single Stage (Call Option)
S = $ 45.00
u = 1.10x
d = 0.90x
X = $ 40.00
i = 5.00%
Freq= 1
Stages= 1
PERIOD 0
PERIOD 1
S = 45.00
Su= 49.50
Sd = 40.50
6.90
Cu= 9.50
Cd= 0.50
p = 0.75
1-p= 0.25
C= 6.90
Example II (Call option w/ no Dividends)
S = $ 45.00
u = 1.10x
d = 0.90x
X = $ 40.00
i = 5.00%
Freq= 1
Stages= 2
PERIOD 0
PERIOD 1
PERIOD 2
S = 45.00
Su= 49.50
8.92
Sd = 40.50
Su^2= 54.45
44.55
Sd^2= 36.45
C1= 11.40
C2= 3.25
Cu^2= 14.45
Cud= 4.55
Cd^2= 0.00
p = 0.75
1-p= 0.25
C= 8.92
Example III (Put Option w/ no Dividends)
S = $ 62.00
u = 1.10x
d = 0.95x
X = $ 70.00
i = 8.00%
Freq= 1
Stages= 2
PERIOD 0
PERIOD 1
PERIOD 2
S = 62.00
Su= 68.20
1.25
Sd = 58.90
Su^2= 75.02
64.79
Sd^2= 55.96
0.64
5.91
Pu^2= 0.00
Pud= 5.21
Pd^2= 14.05
p = 0.87
1-p= 0.13
P= 1.25
Example IV (Call Option w/ Dividends)
S = $ 30.00
u = 1.15x
d = 0.90x
X = $ 25.00
i = 5.00%
Div (δ)= 6.00%
Freq= 1
Stages= 2
PERIOD 0
PERIOD 1
PERIOD 1(x-div)
PERIOD 2
S = 30.00
Su= 34.50
5.84
Sd = 27.00
x-dividend
32.43
25.38
Su^2= 37.29
29.19
29.19
Sd^2= 22.84
8.62
2.39
Cu^2= 12.29
Cud= 4.19
Cd^2= 0.00
p = 0.60
1-p= 0.40
C= 5.84

BLACK-SHOLES OPTION PRICING MODEL



The **Black–Scholes** model is a mathematical description of financial markets and derivative investment instruments. The model develops partial differential equations whose solution, the **Black–Scholes formula**, is widely used in the pricing of European-style options.

Black-Scholes Model - Definition

A mathematical formula designed to price an option as a function of certain variables-generally stock price, striking price, volatility, time to expiration, dividends to be paid, and the current risk-free interest rate.

Black-Scholes Model - Introduction

The Black-Scholes model is a tool for equity options pricing. Prior to the development of the Black-Scholes Model, there was no standard options pricing method and nobody can put a fair price to charge for options. The Black-Scholes Model turned that guessing game into a mathematical science which helped develop the options market into the lucrative industry it is today. Options traders compare the prevailing option price in the exchange against the theoretical value derived by the Black-Scholes Model in order to determine if a particular option contract is over or under valued, hence assisting them in their options trading decision. The Black-Scholes Model was originally created for the pricing and hedging of European Call and Put options as the American Options market, the CBOE, started only 1 month before the creation of the Black-Scholes Model. The difference in the pricing of European options and American options is that options pricing of European options do not take into consideration the possibility of early exercising. American options therefore command a higher price than European options due to the flexibility to exercise the option at anytime. The classic Black-Scholes Model does not take this extra value into consideration in its calculations.

Black-Scholes Model Assumptions

There are several assumptions underlying the Black-Scholes model of calculating options pricing. The most significant is that volatility, a measure of how much a stock can be expected to move in the near-term, is a constant over time. The Black-Scholes model also assumes stocks move in a manner referred to as a random walk; at any given moment, they are as likely to move up as they are to move down. These assumptions are combined with the principle that options pricing should provide no immediate gain to either seller or buyer.

The exact 6 assumptions of the Black-Scholes Model are :

1. Stock pays no dividends

2. Option can only be exercised upon expiration

3. Market direction cannot be predicted, hence "Random Walk"

4. No commissions are charged in the transaction

5. Interest rates remain constant

6. Stock returns are normally distributed, thus volatility is constant over time

As you can see, the validity of many of these assumptions used by the Black-Scholes Model is questionable or invalid, resulting in theoretical values which are not always accurate. Hence, theoretical values derived from the Black-Scholes Model are only good as a guide for relative comparison and is not an exact indication to the over or under priced nature of a stock option.

Model assumptions

The Black–Scholes model of the market for a particular equity makes the following explicit assumptions:

- It is possible to borrow and lend cash at a known constant risk-free interest rate. This restriction has been removed in later extensions of the model.
- The price follows a Geometric Brownian motion with constant drift and volatility. It follows from this that the return is a Log-normal distribution. This often implies the validity of the efficient-market hypothesis.
- There are no transaction costs or taxes.
- The stock does not pay a dividend (see below for extensions to handle dividend payments).
- All securities are perfectly divisible (*i.e.* it is possible to buy any fraction of a share).
- There are no restrictions on short selling.
- There is no arbitrage opportunity
- Options use the European exercise terms, which dictate that options may only be exercised on the day of expiration.

From these conditions in the market for an equity (and for an option on the equity), the authors show that "it is possible to create a hedged position, consisting of a long position in the stock and a short position in [calls on the same stock], whose value will not depend on the price of the stock."[3]

Several of these assumptions of the original model have been removed in subsequent extensions of the model. Modern versions account for changing interest rates (Merton, 1976), transaction costs and taxes (Ingerson, 1976), and dividend payout (Merton, 1973).

The Black Scholes formula calculates the price of European put and call options. It can be obtained by solving the Black–Scholes partial differential equation.

The value of a call option in terms of the Black–Scholes parameters is:

$$C\ (S,t) = SN\ (d_1) – Xe^{–r(T-t)}\ N(d_2)$$

$$d_1 = [\ \ln\ (So/X) + (r + \sigma^2 / 2)\ (T – t)\ ] / [\ \sigma \text{ SQR of } (T – t)\ ]$$

$$d_2 = d_1 – \sigma \text{ SQRT of } (T – t)$$

The price of a put option is:

$$P(S,t) = Xe^{-r(T-t)} - S + (SN(d_1) - Xe^{-r(T-t)} N(d_2)) = Xe^{-r(T-t)} - S + C(S,t)$$

For both, as above:

- N(•) is the cumulative distribution function of the standard normal distribution
- T - t is the time to maturity
- S is the spot price of the underlying asset
- X is the strike price
- r is the risk free rate (annual rate, expressed in terms of continuous compounding)
- σ is the volatility in the log-returns of the underlying

Interpretation

$N(d_1)$ and $N(d_2)$ are the probabilities of the option expiring in-the-money under the equivalent exponential martingale probability measure (numéraire = stock) and the equivalent martingale probability measure (numéraire = risk free asset), respectively. The equivalent martingale probability measure is also called the risk-neutral probability measure. Note that both of these are *probabilities* in a measure theoretic sense, and neither of these is the true probability of expiring in-the-money under the real probability measure. In order to calculate the probability under the real ("physical") probability measure, additional information is require

d—the drift term in the physical measure, or equivalently, the market price of risk.

Example

Suppose you want to value a call option under the following circumstances

Stock Price	$S_0 = 100$
Exercise Price	X=95
Interest Rate	r= .10
Dividend Yield	δ = 0
Time to expiration	T = .25 (one-quarter year)
Standard Deviation	σ = .50

First calculate

d1 = [ln (100/95) + (.10-0 + $.5^2$/2).25] / [.5 SQRT of .25] = .43

d2 = .43 - .5 SQRT of .25 = .18

Next find N (d1) and d N(d2). The normal distribution function is tabulated and may be found in many statistics books. A table of N (d) is provided as Table 16.2 in the book page 521. The normal distribution function N(d) is also provided in any spreadsheet program. In Excel the function name is NORMSDIST, so using EXCEL (using interpolation for 43), we find that

N(.43) = .6664
N(.18) = .5714

Finally, remember that with dividends (δ) = o
$S_0 e^{-\delta T}$ = S0,

Thus the value of the call option is

C = 100 x .6664 – $95e^{-10x0.25}$ x .5714

=66.64 – 52.94 = $13.70

BLACK-SCHOLES OPTION VALUATION METHOD B/S - CALL OPTION

A	B	C	D	E	F	G
5						
6	INPUT				OUTPUT	
7						
8		Standard Deviation (σ) =	0.5		d1 =	0.430
9		Expiration (in years) (T) =	0.25		d2 =	0.180
10		Risk-Free Rate (Annual) (i) =	0.1		N(d1) =	0.666
11		Stock Price (S) =	100		N(d2) =	0.571
12		Exercise Price (X) =	95			
13		Dividend Yield (annual) (δ) =	0		C =	13.6953

Compound at e		
Face Value	$	1.00
Interest		10%
Years		10
Description	**Compound**	**FV**
Annual	1	2.5937425
Semi	2	2.6532977
Quarterly	4	2.6850638
Monthly	12	2.7070415
Daily	365	2.7179096
Hourly	8,760	2.7182663
By Minute	525,600	2.7182816
By Second	31,536,000	2.7182819
Infinite	e	2.7182818

LONG CALCULATION (Break Down Approach)

D1 =	Ln (S / X)	(i-δ+σ^2 /2)	σ√t
D1 =	0.051293294	0.05625	0.25

D1 =	0.43017
N (d1) =	0.66647
D2=	0.18017
N (d2) =	0.57149
C =	13.70

PV calculation using e

e = PV x (1+i)^t
PV = e / (1+i)^t
PV = e ^-it

2. BLACK-SCHOLES OPTION VALUATION METHOD B/S - PUT OPTION

A	B	C	D	E	F	G
32						
33	INPUT				OUTPUT	
34						
35		Standard Deviation (σ) =	0.5		d1 =	0.430
36		Expiration (in years) (T) =	0.25		d2 =	0.180
37		Risk-Free Rate (Annual) (i) =	0.1		N(d1) =	0.666
38		Stock Price (S) =	100		N(d2) =	0.571
39		Exercise Price (X) =	95			
40		Dividend Yield (annual) (δ) =	0		P =	6.3497

Compound at e		
Face Value	$	1.00
Interest		10%
Years		10
Description	**Compound**	**FV**
Annual	1	2.5937425
Semi	2	2.6532977
Quarterly	4	2.6850638
Monthly	12	2.7070415
Daily	365	2.7179096
Hourly	8,760	2.7182663
By Minute	525,600	2.7182816
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Infinite	e	2.7182818

LONG CALCULATION (Break Down Approach)

D1 =	Ln (S / X)	(i-δ+σ^2 /2)	σ√t
D1 =	0.051293294	0.05625	0.25

D1 =	0.430173178
N (d1) =	0.666465164
D2=	0.180173178
N (d2) =	0.571491692
P =	6.3497

PV calculation using e

e = PV x (1+i)^t
PV = e / (1+i)^t
PV = e ^-it

3. PUT-CALL PARITY METHOD FOR CALCULATING THE PUT OPTION KNOWING THE CALL PRICE (same data as above)

C - P = S - X .e^ -it
P = X.e^-it - S + C

P =	6.3497

Review – Options

$C = S\,e^{-\delta T}\,N(d1) - X\,e^{-iT}\,N(d2)$

$P = X\,e^{-iT}\,(1 - N(d2)) - S\,e^{-\delta T}\,(1 - N(d1))$

Volatility is the question on the B/S –which assumes constant SD throughout the exercise period - The time series of implied volatility

THE PUT – CALL PARITY RELATIONSHIP

- Put prices can be derived simply from the prices of call
- European Put or Call options are linked together in an equation known as the Put-Call parity relationship

	St <= X	**St > X**
Payoff of Call Held	0	St - X
Payoff of Put Written	-(X – St)	0
Total	St – X	St – X

$PV(x) = X\,e^{-rt}$

The option has a payoff identical to that of the leveraged equity position; the costs of establishing them must be equal

- C – P Cost of Call purchased = Premium received from Put written

- The leverage Equity position requires a net cash outlay of $S - X\,e^{-rt}$ the Cost of the stock less the process from borrowing

- $C - P = S - X\,e^{-rt}$ PUT-CALL Parity Relationship - proper relationship between Call and Put

Example 16.3

S = $110
C = $14 for 6 months with X = $105
P = $5 for 6 months with X=$105
rf = 5.0% (continuously compounding at e)

Assumptions:

$C – P = S – X e^{–rT}$?????
$14 – 5 = 110 – 105 .e^{– 0.5 x 0.5}$
9 = 7.59
This a violation of parity…. Indicates mispricing and leads to Arbitrage Opportunity

You can buy relatively cheap portfolio (buy the stock plus borrowing position represented on the right side of the equation and .sell the expensive portfolio

STRATEGY – In six months the stock will be worth Sr, so you borrow PV of X ($105) and pay back the loan with interest resulting in cash outflow of $105

Sr – 105 writing the call if Sr exceeds 105
Purchase Puts will pay 105 – Sr if the stock is below the $105

	Strategy	Immediate CF	CF if Sr < 105	CF if Sr > 105
1	Buy Stock	-110.00	Sr	Sr
2	Borrow $Xe^{–iT}$ = $102.41	+102.41	-105	-105
3	Sell Call	14.00	0	-(Sr – 105)
4	Buy Call	-5.00	105 – Sr	0
		1,41	0	0

Whish is the difference of between 9.00 and 7.59 – riskless return

This applies if No dividends and under the European option

If Dividend then

P = C – S + PV (X) + PV (Dividend) …. Representing that the Dividend (δ) is paid during the life of the option.

<u>Example</u>

Using the IBM example – today is February 6

X = $100 (March calls)
T = 42 days
C = $2.80
P = $6.47
S = 96.14
I = 2.0%
δ = 0

P = C – S + PV (X) + PV (Dividend) or P = C + PV (X) – S + PV (δ)

???? $6.47 = 2.80 + 100 / (1+0.02)^{42/365} - 96.14 + 0$

6.47 = 6.63 is not that valuable to go after the reprising arbitrage

PUT OPTION VALUATION

$P = X e^{-iT} (1- N (d2)) - S e^{-\delta T} (1 - N (d1))$

Using the data from previous example

$P = 95. e^{-10x.25} (1 - 0.05714) - 100 (1 - 0.6664)$

P = 6.35

PUT-CALL Parity

P = C + PV (X) – So + PV (Div)

\P = $13.70 + 95 . e^{-10 X 0.25} - 100 + 0$
<u>Hedge Ratios & the B/S format</u>

The Hedge ratio is commonly called the **Option Delta**. Is the change in the price of call option for $1 increased in the stock price

This is the slope of value function evaluated at the current stock price

For Example

Slope of the curve at S = $120 equals .60. As the stock increases by $1, the option increase on 0.60

For every Call Option Written, .60 shares of stock would be needed to hedge the Investment portfolio.

For example, if one writes 10 options and holds 6 shares of stock,

H = .60 ……… a $1 increase in stock will result $6 gain ($1x 6 shares) and with the loss of $6 on 10 options written (10 x $0.60)

- The Hedge Ratio for a Call is N (d1),
- with the hedge ratio for a Put [N (d1) – 1]
- N (d) is the area under standard deviation (normal)
- Therefore, the Call option Hedge Ratio must be positive and less than 1.0
- And the Put option Hedge Ratio is negative and less than 1.0

Example 16.5

2 Portfolios

Portfolio	A	B
BUY	750 IBM Calls 200 Shares of IBM	800 shares of IBM

Which portfolio has a greater dollar exposure to IBM price movement?

Using the Hedge ratio you could answer that question:

Each Option change in value by H dollars for each $1 change in stock price

If H = 0.6, then 750 options = equivalent 450 shares (0.6 x 750)

Portfolio A = 450 equivalent + 200 shares which is less than Portfolio B with 800 shares